Noncommutative sedeons and their application in field theory

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(submitted 17 November 2011)

We present sixteen-component values “sedeons”, generating associative noncommutative space-time algebra. The generalized field equations based on sedeonic potentials and space-time operators are proposed. It is shown that the sedeonic second-order wave equation for massive field can be represented in the form of the system of Maxwell-like first order equations. The sedeonic generalization of first order Dirac equation for massive and massless fields is also discussed.

PACS numbers: 03.65.-w, 03.65.Fd, 03.65.Pm, 02.10.De.

1. Introduction

The application of hypercomplex numbers and multivectors in the field theory has a long history. In particular, the simplest generalization of electrodynamics and quantum mechanics was developed on the basis of quaternions [1-5]. The structure of quaternions with four components (scalar and vector) corresponds to the relativistic four-vector approach that allows one to reformulate field equations in terms of quaternionic algebra. However, the essential imperfection of the quaternionic algebra is that the quaternions do not include pseudoscalar and pseudovector components. The consideration of total space symmetry with respect to spatial inversion leads us to the eight-component structures enclosing scalar, pseudoscalar, vector and pseudovector components. However, attempts to apply different eight-component values such as biquaternions, octonions [6-11] and multivectors generating associative Clifford algebras [12] have not made appreciable progress. In particularly, the few attempts to describe relativistic particles by means of octonion wave functions are confronted by difficulties connected with octonions nonassociativity [10]. On the other hand, a consistent relativistic approach requires taking into consideration full time and space symmetries that leads to the sixteen-component space-time algebras.

There are a few approaches in the development of sixteen-component field theory. One of them is the application of hypernumbers sedenions, which are obtained from octonions by Cayley-Dickson extension procedure [13-16]. But as in the case of octonions the essential imperfection of sedenions is their nonassociativity. Another approach is based on application of hypercomplex multivectors generating associative space-time Clifford algebras. The basic idea of such multivectors is an introduction of additional noncommutative time unit vector, which is orthogonal to the space unit vectors [17, 18]. However, the application of such multivectors in quantum mechanics is considered in general as one of abstract algebraic scheme enabling the reformulation of Dirac equation for the multicomponent wave functions but does not touch the physical entity of this equation.

Recently we have developed an alternative approach based on our scalar-vector conception [19-22] realized in eight-component octons and sixteen-component sedeons. In particular, in Ref. 22 we considered a variant of sixteen-component sedeonic space-time Clifford algebra with noncommutative vector basis and commutative space-time units that allowed us to reformulate the field equations in terms of scalar-vector field potentials. However, these equations have some asymmetry and contain the special non-sedeonic operators of space-time reflection [22]. In this paper we present a new version of the sedeonic space-time algebra with non-commutative bases and demonstrate some of its application to the symmetric reformulation of the basic field theory equations.

2. Sedeonic space-time algebra

The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

(1) Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.

(2) Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.

(3) Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.

(4) Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

The indexes **t** and **r** indicate the transformations (**t** for time inversion and **r** for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time object sedeon $\tilde{\boldsymbol{V}}$, which is defined by the following expression:

$$\tilde{\boldsymbol{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \tag{1}$$

Let us introduce scalar-vector basis $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, where value $\mathbf{a_0} \equiv 1$ is absolute scalar unit and the values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are absolute unit vectors generating the right Cartesian basis. We introduce also four space-time scalar units $\mathbf{e_0}$, $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, where value $\mathbf{e_0} \equiv 1$ is a absolute scalar unit; $\mathbf{e_1} \equiv \mathbf{e_t}$ is a time scalar unit; $\mathbf{e_2} \equiv \mathbf{e_r}$ is a space scalar unit; $\mathbf{e_3} \equiv \mathbf{e_{tr}}$ is a space-time scalar unit. Using scalar-vector basis $\mathbf{a_k}$ (k = 0, 1, 2, 3) and space-time scalar units $\mathbf{e_n}$ (n = 0, 1, 2, 3) we can introduce unified sedeonic components V_{nk} in accordance with the following relations

$$\begin{aligned}
V &= \mathbf{e_0} V_{00} \mathbf{a_0} , \\
\vec{V} &= \mathbf{e_0} \left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right), \\
V_{\mathbf{t}} &= \mathbf{e_1} V_{10} \mathbf{a_0} , \\
\vec{V}_{\mathbf{t}} &= \mathbf{e_1} \left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right), \\
V_{\mathbf{r}} &= \mathbf{e_2} V_{20} \mathbf{a_0} , \\
\vec{V}_{\mathbf{r}} &= \mathbf{e_2} \left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right), \\
V_{\mathbf{rt}} &= \mathbf{e_3} V_{30} \mathbf{a_0} , \\
\vec{V}_{\mathbf{rt}} &= \mathbf{e_3} \left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{2}$$

Then sedeon (1) can be written in the following expanded form:

$$\begin{aligned}
\tilde{\boldsymbol{V}} = {} & \mathbf{e_0} \left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) + \mathbf{e_1} \left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) \\
& + \mathbf{e_2} \left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right) + \mathbf{e_3} \left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{3}$$

The sedeonic components V_{nk} are numbers (complex in general). Further we will use symbol 1 instead units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and space-time units $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are presented in tables 1 and 2.

Table 1.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$
$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$
$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1

In the tables and further the value i is the imaginary unit ($i^2 = -1$). Note that sedeonic units $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ generate the anticommutative algebras, but $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ commute with $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Thus the sedeon $\tilde{\boldsymbol{V}}$ is the complicated space-time object consisting of absolute scalar, space scalar, time scalar, space-time scalar, absolute vector, space vector, time vector and space-time vector.

Introducing designations of scalar-vector values

$$\begin{aligned}
\bar{\boldsymbol{V}}_0 &= V_{00} + V_{01}\mathbf{a}_1 + V_{02}\mathbf{a}_2 + V_{03}\mathbf{a}_3 , \\
\bar{\boldsymbol{V}}_1 &= V_{10} + V_{11}\mathbf{a}_1 + V_{12}\mathbf{a}_2 + V_{13}\mathbf{a}_3 , \\
\bar{\boldsymbol{V}}_2 &= V_{20} + V_{21}\mathbf{a}_1 + V_{22}\mathbf{a}_2 + V_{23}\mathbf{a}_3 , \\
\bar{\boldsymbol{V}}_3 &= V_{30} + V_{31}\mathbf{a}_1 + V_{32}\mathbf{a}_2 + V_{33}\mathbf{a}_3 ,
\end{aligned} \tag{4}$$

we can write the sedeon (3) in the following compact form

$$\tilde{\boldsymbol{V}} = \bar{\boldsymbol{V}}_0 + \mathbf{e}_1\bar{\boldsymbol{V}}_1 + \mathbf{e}_2\bar{\boldsymbol{V}}_2 + \mathbf{e}_3\bar{\boldsymbol{V}}_3 . \tag{5}$$

On the other hand, introducing designations of space-time sedeon-scalars

$$\begin{aligned}
\boldsymbol{V}_0 &= (V_{00} + \mathbf{e}_1 V_{10} + \mathbf{e}_2 V_{20} + \mathbf{e}_3 V_{30}) , \\
\boldsymbol{V}_1 &= (V_{01} + \mathbf{e}_1 V_{11} + \mathbf{e}_2 V_{21} + \mathbf{e}_3 V_{31}) , \\
\boldsymbol{V}_2 &= (V_{02} + \mathbf{e}_1 V_{12} + \mathbf{e}_2 V_{22} + \mathbf{e}_3 V_{32}) , \\
\boldsymbol{V}_3 &= (V_{03} + \mathbf{e}_1 V_{13} + \mathbf{e}_2 V_{23} + \mathbf{e}_3 V_{33}) ,
\end{aligned} \tag{6}$$

we can write the sedeon (3) in the compact form

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0 + \boldsymbol{V}_1\mathbf{a}_1 + \boldsymbol{V}_2\mathbf{a}_2 + \boldsymbol{V}_3\mathbf{a}_3 , \tag{7}$$

or introducing the sedeon-vector

$$\vec{\boldsymbol{V}} = \vec{V} + \vec{V}_{\mathbf{t}} + \vec{V}_{\mathbf{r}} + \vec{V}_{\mathbf{tr}} = \boldsymbol{V}_1\mathbf{a}_1 + \boldsymbol{V}_2\mathbf{a}_2 + \boldsymbol{V}_3\mathbf{a}_3 , \tag{8}$$

it can be represented in following compact form

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0 + \vec{\boldsymbol{V}} . \tag{9}$$

Further we will indicate sedeon-scalars and sedeon-vectors with the bold capital letters.

Let us consider the sedeonic multiplication in detail. The sedeonic product of two sedeons $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ can be represented in the following form

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} = \left(\boldsymbol{A}_0 + \vec{\boldsymbol{A}}\right)\left(\boldsymbol{B}_0 + \vec{\boldsymbol{B}}\right) = \boldsymbol{A}_0\boldsymbol{B}_0 + \boldsymbol{A}_0\vec{\boldsymbol{B}} + \vec{\boldsymbol{A}}\boldsymbol{B}_0 + \left(\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}\right) + \left[\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}\right] \tag{10}$$

Here we denoted the sedeonic scalar multiplication of two sedeon-vectors (internal product) by symbol “$\cdot$” and round brackets

$$\left(\vec{A}\cdot\vec{B}\right) = A_1B_1 + A_2B_2 + A_3B_3\,, \tag{11}$$

and sedeonic vector multiplication (external product) by symbol “×” and square brackets,

$$\left[\vec{A}\times\vec{B}\right] = i\left(A_2B_3 - A_3B_2\right)\mathbf{a_1} + i\left(A_3B_1 - A_1B_3\right)\mathbf{a_2} + i\left(A_1B_2 - A_2B_1\right)\mathbf{a_3}\,. \tag{12}$$

In (11) and (12) the multiplication of sedeonic components is performed in accordance with (6) and table 2. Note that in sedeonic algebra the vector triple product has some difference from Gibbs vector algebra. Let us consider three absolute vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Then the formula for the vector triple product in sedeonic algebra has the following form:

$$\left[\vec{A}\times\left[\vec{B}\times\vec{C}\right]\right] = -\vec{B}\left(\vec{A}\cdot\vec{C}\right) + \vec{C}\left(\vec{A}\cdot\vec{B}\right). \tag{13}$$

Thus the sedeonic product

$$\tilde{F} = \tilde{A}\tilde{B} = F_0 + \vec{F} \tag{14}$$

has the following components:

$$\begin{aligned} F_0 &= A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3\,, \\ F_1 &= A_1B_0 + A_0B_1 + i\left(A_2B_3 - A_3B_2\right), \\ F_2 &= A_2B_0 + A_0B_2 + i\left(A_3B_1 - A_1B_3\right), \\ F_3 &= A_3B_0 + A_0B_3 + i\left(A_1B_2 - A_2B_1\right). \end{aligned} \tag{15}$$

3. Sedeonic space rotation and space-time inversion

The rotation of sedeon $\tilde{V}$ on the angle θ around the absolute unit vector $\vec{n}$ is realized by uncompleted sedeon

$$\tilde{U} = \cos\left(\theta/2\right) + i\sin\left(\theta/2\right)\vec{n} \tag{16}$$

and by sedeon $\tilde{U}^*$ complex conjugated to $\tilde{U}$:

$$\tilde{U}^* = \cos\left(\theta/2\right) - i\sin\left(\theta/2\right)\vec{n} \tag{17}$$

with

$$\tilde{U}^*\tilde{U} = \tilde{U}\tilde{U}^* = 1\,. \tag{18}$$

The transformed sedeon $\tilde{V}'$ is defined as sedeonic product

$$\tilde{V}' = \tilde{U}^*\,\tilde{V}\;\tilde{U}\,, \tag{19}$$

Thus the transformed sedeon $\tilde{V}'$ can be written as

$$\begin{aligned} \tilde{V}' &= \left[\cos\left(\theta/2\right) - i\sin\left(\theta/2\right)\vec{n}\right]\left(V_0 + \vec{V}\right)\left[\cos\left(\theta/2\right) + i\sin\left(\theta/2\right)\vec{n}\right] = \\ &= V_0 + \vec{V}\cos\theta + \left(1-\cos\theta\right)\left(\vec{n}\cdot\vec{V}\right)\vec{n} - i\sin\theta\left[\vec{n}\times\vec{V}\right]. \end{aligned} \tag{20}$$

It is clearly seen that rotation does not transform the sedeon-scalar part, but sedeonic vector $\vec{V}$ is rotated on the angle θ around $\vec{n}$.

The operations of time inversion ($\hat{R}_{\mathbf{t}}$), space inversion ($\hat{R}_{\mathbf{r}}$) and space-time inversion ($\hat{R}_{\mathbf{tr}}$) are connected with transformations in $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ basis and can be represented as

$$\hat{R}_{\mathrm{t}}\widetilde{V} = \mathbf{e}_2\widetilde{V}\mathbf{e}_2 = \overline{V}_0 - \mathbf{e}_1\overline{V}_1 + \mathbf{e}_2\overline{V}_2 - \mathbf{e}_3\overline{V}_3 ,$$

$$\hat{R}_{\mathrm{r}}\widetilde{V} = \mathbf{e}_1\widetilde{V}\mathbf{e}_1 = \overline{V}_0 + \mathbf{e}_1\overline{V}_1 - \mathbf{e}_2\overline{V}_2 - \mathbf{e}_3\overline{V}_3 , \tag{21}$$

$$\hat{R}_{\mathbf{tr}}\widetilde{V} = \mathbf{e}_3\widetilde{V}\mathbf{e}_3 = \overline{V}_0 - \mathbf{e}_1\overline{V}_1 - \mathbf{e}_2\overline{V}_2 + \mathbf{e}_3\overline{V}_3 .$$

4 Sedeonic Lorentz transformations

The relativistic event four-vector can be represented in the follow sedeonic form:

$$\tilde{\boldsymbol{S}} = i\mathbf{e}_1 ct + \mathbf{e}_2\vec{r} . \tag{22}$$

The square of this value is the Lorentz invariant

$$\tilde{\boldsymbol{S}}\ \tilde{\boldsymbol{S}} = -c^2t^2 + x^2 + y^2 + z^2 . \tag{23}$$

The Lorentz transformation of event four-vector can be realized by uncompleted sedeons

$$\tilde{\boldsymbol{L}} = \mathrm{ch}\,\vartheta - \mathbf{e}_3\vec{m}\ \mathrm{sh}\,\vartheta , \tag{24}$$

$$\tilde{\boldsymbol{L}}^* = \mathrm{ch}\,\vartheta + \mathbf{e}_3\vec{m}\ \mathrm{sh}\,\vartheta , \tag{25}$$

where $\mathrm{th}\,2\vartheta = \mathrm{v}/c$, v is velocity of motion along the absolute unit vector $\vec{m}$. Note that

$$\tilde{\boldsymbol{L}}^*\tilde{\boldsymbol{L}} = \tilde{\boldsymbol{L}}\ \tilde{\boldsymbol{L}}^* = 1 . \tag{26}$$

The transformed event four-vector $\tilde{\boldsymbol{S}}'$ is written as

$$\begin{aligned}\tilde{\boldsymbol{S}}' = \tilde{\boldsymbol{L}}^*\tilde{\boldsymbol{S}}\ \tilde{\boldsymbol{L}} = \left(\mathrm{ch}\,\vartheta + \mathbf{e}_3\mathrm{sh}\,\vartheta\,\vec{m}\right)\left(i\mathbf{e}_1 ct + \mathbf{e}_2\vec{r}\right)\left(\mathrm{ch}\,\vartheta - \mathbf{e}_3\mathrm{sh}\,\vartheta\,\vec{m}\right) = \\ i\mathbf{e}_1 ct\ \mathrm{ch}2\vartheta - i\mathbf{e}_1\left(\vec{m}\cdot\vec{r}\right)\mathrm{sh}2\vartheta \\ +\mathbf{e}_2\vec{r}\ \mathrm{ch}^2\vartheta - \mathbf{e}_2 ct\vec{m}\ \mathrm{sh}2\vartheta + \mathbf{e}_2\left(\vec{m}\cdot\vec{r}\right)\vec{m}\ \mathrm{sh}^2\vartheta + \mathbf{e}_2\left[\left[\vec{m}\times\vec{r}\right]\times\vec{m}\right]\mathrm{sh}^2\vartheta .\end{aligned} \tag{27}$$

Separating the values with $\mathbf{e}_1$ and $\mathbf{e}_2$ we get the well known formulas for time and coordinates transformation [23]:

$$t' = \frac{t - x\,\mathrm{v}/c^2}{\sqrt{1-\mathrm{v}^2/c^2}} ,\ x' = \frac{x - t\,\mathrm{v}}{\sqrt{1-\mathrm{v}^2/c^2}} ,\ y' = y ,\ z' = z , \tag{28}$$

where x is the coordinate along the $\vec{m}$ vector.

Let us also consider the Lorentz transformation of full sedeon $\tilde{\boldsymbol{V}}$.

The transformed sedeon $\tilde{\boldsymbol{V}}'$ can be written as sedeonic product

$$\tilde{\boldsymbol{V}}' = \tilde{\boldsymbol{L}}^*\ \tilde{\boldsymbol{V}}\ \tilde{\boldsymbol{L}} . \tag{29}$$

$$\begin{aligned}\tilde{\boldsymbol{V}}' &= \left(\mathrm{ch}\,\vartheta + \mathbf{e}_{\mathbf{tr}}\mathrm{sh}\,\vartheta\,\vec{m}\right)\left(\boldsymbol{V}_0 + \vec{\boldsymbol{V}}\right)\left(\mathrm{ch}\,\vartheta - \mathbf{e}_{\mathbf{tr}}\mathrm{sh}\,\vartheta\,\vec{m}\right) \\ &= \boldsymbol{V}_0\mathrm{ch}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\boldsymbol{V}_0\mathbf{e}_{\mathbf{rt}}\mathrm{sh}^2\vartheta + \left(\mathbf{e}_{\mathbf{tr}}\boldsymbol{V}_0 - \boldsymbol{V}_0\mathbf{e}_{\mathbf{tr}}\right)\vec{m}\ \mathrm{ch}\,\vartheta\ \mathrm{sh}\,\vartheta \\ &+ \vec{\boldsymbol{V}}\mathrm{ch}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\vec{m}\,\vec{\boldsymbol{V}}\vec{m}\ \mathbf{e}_{\mathbf{tr}}\mathrm{sh}^2\vartheta + \left(\mathbf{e}_{\mathbf{tr}}\vec{m}\,\vec{\boldsymbol{V}} - \vec{\boldsymbol{V}}\vec{m}\mathbf{e}_{\mathbf{tr}}\right)\mathrm{ch}\,\vartheta\ \mathrm{sh}\,\vartheta .\end{aligned} \tag{30}$$

Rewriting the expression (30) with scalar (11) and vector (12) products we get

$$\tilde{\boldsymbol{V}}' = \boldsymbol{V}_0 \mathrm{ch}^2\vartheta - \mathbf{e}_{\mathbf{tr}}\boldsymbol{V}_0\mathbf{e}_{\mathbf{tr}}\mathrm{sh}^2\vartheta + \left(\mathbf{e}_{\mathbf{tr}}\boldsymbol{V}_0 - \boldsymbol{V}_0\mathbf{e}_{\mathbf{tr}}\right)\vec{m}\,\mathrm{ch}\vartheta\,\mathrm{sh}\vartheta$$
$$+\vec{\boldsymbol{V}}\mathrm{ch}^2\vartheta + \mathbf{e}_{\mathbf{tr}}\vec{\boldsymbol{V}}\mathbf{e}_{\mathbf{tr}}\mathrm{sh}^2\vartheta - 2\mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{\boldsymbol{V}}\right)\mathbf{e}_{\mathbf{tr}}\vec{m}\,\mathrm{sh}^2\vartheta \tag{31}$$
$$+\left(\mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{\boldsymbol{V}}\right)-\left(\vec{\boldsymbol{V}}\cdot\vec{m}\right)\mathbf{e}_{\mathbf{tr}}\right)\mathrm{ch}\vartheta\,\mathrm{sh}\vartheta + \left(\mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{\boldsymbol{V}}\right]-\left[\vec{\boldsymbol{V}}\times\vec{m}\right]\mathbf{e}_{\mathbf{tr}}\right)\mathrm{ch}\vartheta\,\mathrm{sh}\vartheta.$$

Thus, the transformed sedeon have the following components:

$$\begin{aligned}
&V' = V\,,\\
&V'_{\mathbf{tr}} = V_{\mathbf{tr}}\,,\\
&V'_{\mathbf{r}} = V_{\mathbf{r}}\,\mathrm{ch}\,2\vartheta + \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\mathrm{sh}\,2\vartheta\,,\\
&V'_{\mathbf{t}} = V_{\mathbf{t}}\,\mathrm{ch}\,2\vartheta + \mathbf{e}_{\mathbf{tr}}\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\mathrm{sh}\,2\vartheta\,,\\
&\vec{V}' = \vec{V}\,\mathrm{ch}\,2\vartheta - 2\left(\vec{m}\cdot\vec{V}\right)\vec{m}\,\mathrm{sh}^2\vartheta + \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}_{\mathbf{rt}}\right]\mathrm{sh}2\vartheta\,,\\
&\vec{V}'_{\mathbf{tr}} = \vec{V}_{\mathbf{tr}}\,\mathrm{ch}\,2\vartheta - 2\left(\vec{m}\cdot\vec{V}_{\mathbf{tr}}\right)\vec{m}\,\mathrm{sh}^2\vartheta + \mathbf{e}_{\mathbf{tr}}\left[\vec{m}\times\vec{V}\right]\mathrm{sh}2\vartheta\,,\\
&\vec{V}'_{\mathbf{r}} = \vec{V}_{\mathbf{r}} + 2\left(\vec{m}\cdot\vec{V}_{\mathbf{r}}\right)\vec{m}\,\mathrm{sh}^2\vartheta + \mathbf{e}_{\mathbf{tr}}V_{\mathbf{t}}\,\vec{m}\,\mathrm{sh}\,2\vartheta\,,\\
&\vec{V}'_{\mathbf{t}} = \vec{V}_{\mathbf{t}} + 2\left(\vec{m}\cdot\vec{V}_{\mathbf{t}}\right)\vec{m}\,\mathrm{sh}^2\vartheta + \mathbf{e}_{\mathbf{tr}}V_{\mathbf{r}}\,\vec{m}\,\mathrm{sh}\,2\vartheta\,.
\end{aligned} \tag{32}$$

It is seen that the sedeon components, which commute with $\mathbf{e}_{\mathbf{rt}}$ are transformed as field intensities while the components that anticommute with $\mathbf{e}_{\mathbf{rt}}$ are transformed as potentials [23].

5. Generalized sedeonic wave equation for massive field

Let us consider the field potential in the form of space-time sedeon

$$\tilde{\boldsymbol{W}}(t,\vec{r}) = \boldsymbol{W}_0(t,\vec{r}) + \vec{\boldsymbol{W}}(t,\vec{r})\,. \tag{33}$$

The potential of free field should satisfy an equation, which is obtained from the Einstein relation for energy and momentum

$$E^2 - c^2p^2 = m^2c^4 \tag{34}$$

by means of changing classical energy E and momentum $\vec{p}$ on corresponding quantum-mechanical operators:

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \text{ and } \hat{p} = -i\hbar\vec{\nabla}\,. \tag{35}$$

The Einstein relation (34) can be written using sedeonic algebra in the following form:

$$\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p} + \mathbf{e}_{\mathbf{tr}}mc^2\right)\left(i\mathbf{e}_{\mathbf{t}}E + \mathbf{e}_{\mathbf{r}}c\vec{p} + \mathbf{e}_{\mathbf{tr}}mc^2\right) = 0\,. \tag{36}$$

Then the generalized sedeonic wave equation for free massive field can be written in the symmetric form

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}}\frac{mc}{\hbar}\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla} - i\mathbf{e}_{\mathbf{tr}}\frac{mc}{\hbar}\right)\tilde{\boldsymbol{W}} = 0\,. \tag{37}$$

Redefining the operators

$$\partial_{\mathbf{t}} = \mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t}\,,$$

$$\vec{\nabla}_{\mathbf{r}} = \mathbf{e}_{\mathbf{r}}\vec{\nabla} = \mathbf{e}_{\mathbf{r}}\left(\frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3\right), \quad (38)$$

$$m_{\mathbf{tr}} = \mathbf{e}_{\mathbf{tr}}\frac{mc}{\hbar}$$

we can rewrite the equation (37) in compact form:

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\tilde{\boldsymbol{W}} = 0 . \quad (39)$$

Let us also consider the generalized sedeonic source

$$\tilde{\boldsymbol{J}}(t,\vec{r}) = \boldsymbol{J}_0(t,\vec{r}) + \vec{\boldsymbol{J}}(t,\vec{r}) . \quad (40)$$

Then nonhomogeneous sedeonic wave equation for massive field can be written in the following form:

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\tilde{\boldsymbol{W}} = \tilde{\boldsymbol{J}} . \quad (41)$$

The sedeonic equation (41) can be represented in the form of system of Maxwell-like first-order equations. Let us consider the sequential action of operators in (41). After the action of the first operator we obtain

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{rt}}\right)\tilde{\boldsymbol{W}} = i\partial_{\mathbf{t}}\boldsymbol{W}_0 + i\partial_{\mathbf{t}}\vec{\boldsymbol{W}}$$
$$-\vec{\nabla}_{\mathbf{r}}\boldsymbol{W}_0 - \left(\vec{\nabla}_{\mathbf{r}}\cdot\vec{\boldsymbol{W}}\right) - \left[\vec{\nabla}_{\mathbf{r}}\times\vec{\boldsymbol{W}}\right] - im_{\mathbf{tr}}\boldsymbol{W}_0 - im_{\mathbf{tr}}\vec{\boldsymbol{W}} . \quad (42)$$

Introducing the scalar and vector field’s intensities

$$\boldsymbol{E}_0 = i\partial_{\mathbf{t}}\boldsymbol{W}_0 - \left(\vec{\nabla}_{\mathbf{r}}\cdot\vec{\boldsymbol{W}}\right) - im_{\mathbf{tr}}\boldsymbol{W}_0 , \quad (43)$$

$$\vec{\boldsymbol{E}} = i\partial_{\mathbf{t}}\vec{\boldsymbol{W}} - \vec{\nabla}_{\mathbf{r}}\boldsymbol{W}_0 - im_{\mathbf{tr}}\vec{\boldsymbol{W}} - \left[\vec{\nabla}_{\mathbf{r}}\times\vec{\boldsymbol{W}}\right], \quad (44)$$

the relation (42) is presented as

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\tilde{\boldsymbol{W}} = \boldsymbol{E}_0 + \vec{\boldsymbol{E}} . \quad (45)$$

Then the wave equation (41) can be written as

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\left(\boldsymbol{E}_0 + \vec{\boldsymbol{E}}\right) = \boldsymbol{J}_0 + \vec{\boldsymbol{J}} . \quad (46)$$

Applying the operator $\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)$ to both parts of equation (46) and separating sedeon-scalar and sedeon-vector parts we get the wave equations for the field intensities

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\boldsymbol{E}_0 = i\partial_{\mathbf{t}}\boldsymbol{J}_0 - \left(\vec{\nabla}_{\mathbf{r}}\cdot\vec{\boldsymbol{J}}\right) - im_{\mathbf{tr}}\boldsymbol{J}_0 , \quad (47)$$

$$\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\left(i\partial_{\mathbf{t}} - \vec{\nabla}_{\mathbf{r}} - im_{\mathbf{tr}}\right)\vec{\boldsymbol{E}} = i\partial_{\mathbf{t}}\vec{\boldsymbol{J}} - \vec{\nabla}_{\mathbf{r}}\boldsymbol{J}_0 - im_{\mathbf{tr}}\vec{\boldsymbol{J}} - \left[\vec{\nabla}_{\mathbf{r}}\times\vec{\boldsymbol{J}}\right]. \quad (48)$$

On the other hand, performing sedeonic multiplication in expression (46) and separating sedeon-scalar and sedeon-vector parts we obtain the system of the first-order equations for the field's intensities:

$$i\partial_{\mathbf{t}}\boldsymbol{E}_0 - \left(\vec{\nabla}_{\mathbf{r}}\cdot\vec{\boldsymbol{E}}\right) - im_{\mathbf{tr}}\boldsymbol{E}_0 = \boldsymbol{J}_0 , \quad (49)$$

$$i\partial_{\mathbf{t}}\vec{\boldsymbol{E}} - \left[\vec{\nabla}_{\mathbf{r}}\times\vec{\boldsymbol{E}}\right] - \vec{\nabla}_{\mathbf{r}}\boldsymbol{E}_0 - im_{\mathbf{tr}}\vec{\boldsymbol{E}} = \vec{\boldsymbol{J}} . \quad (50)$$

In special case with the mass equal to zero the equations (49) and (50) coincide with the Maxwell equations for electromagnetic field in a vacuum [22]. Indeed, choosing sedeonic potential as

$$\tilde{\boldsymbol{W}} = i\mathbf{e}_{\mathbf{t}}\varphi + \mathbf{e}_{\mathbf{r}}\vec{A}, \tag{51}$$

and the source of electromagnetic field as

$$\tilde{\boldsymbol{J}} = -i\mathbf{e}_{\mathbf{t}}4\pi\rho - \mathbf{e}_{\mathbf{r}}\frac{4\pi}{c}\vec{j}\,, \tag{52}$$

we get the following wave equation:

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}\varphi + \mathbf{e}_{\mathbf{r}}\vec{A}\right) = -i\mathbf{e}_{\mathbf{t}}4\pi\rho - \mathbf{e}_{\mathbf{r}}\frac{4\pi}{c}\vec{j}\,. \tag{53}$$

After the action of the first operator we obtain

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}\varphi + \mathbf{e}_{\mathbf{r}}\vec{A}\right) = -\frac{1}{c}\frac{\partial\varphi}{\partial t} - \mathbf{e}_{\mathbf{tr}}\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \mathbf{e}_{\mathbf{tr}}\vec{\nabla}\varphi - \left(\vec{\nabla}\cdot\vec{A}\right) - \left[\vec{\nabla}\times\vec{A}\right]. \tag{54}$$

Using the sedeonic definitions of the electric and magnetic fields

$$\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi, \\ \vec{H} &= -i\left[\vec{\nabla}\times\vec{A}\right] \end{aligned} \tag{55}$$

and taking into account Lorentz gauge condition

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \left(\vec{\nabla}\cdot\vec{A}\right) = 0\,, \tag{56}$$

the expression (54) can be rewritten as

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(i\mathbf{e}_{\mathbf{t}}\varphi + \mathbf{e}_{\mathbf{r}}\vec{A}\right) = \mathbf{e}_{\mathbf{tr}}\vec{E} - i\vec{H}\,. \tag{57}$$

Then the wave equation (53) can be represented in the following form:

$$\left(i\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_{\mathbf{r}}\vec{\nabla}\right)\left(\mathbf{e}_{\mathbf{tr}}\vec{E} - i\vec{H}\right) = -i\mathbf{e}_{\mathbf{t}}4\pi\rho - \mathbf{e}_{\mathbf{r}}\frac{4\pi}{c}\vec{j}\,. \tag{58}$$

Performing sedeonic multiplication in the left part of equation (58) we get

$$\begin{aligned} &\mathbf{e}_{\mathbf{r}}\frac{1}{c}\frac{\partial\vec{E}}{\partial t} - i\mathbf{e}_{\mathbf{t}}\left(\vec{\nabla}\cdot\vec{E}\right) - i\mathbf{e}_{\mathbf{t}}\left[\vec{\nabla}\times\vec{E}\right] \\ &+\mathbf{e}_{\mathbf{t}}\frac{1}{c}\frac{\partial\vec{H}}{\partial t} + i\mathbf{e}_{\mathbf{r}}\left(\vec{\nabla}\cdot\vec{H}\right) + i\mathbf{e}_{\mathbf{r}}\left[\vec{\nabla}\times\vec{H}\right] = -i\mathbf{e}_{\mathbf{t}}4\pi\rho - \mathbf{e}_{\mathbf{r}}\frac{4\pi}{c}\vec{j}. \end{aligned} \tag{59}$$

Separating space-time values we obtain the system of Maxwell equations in the following form:

$$\begin{aligned}
&\mathbf{e}_\mathbf{t}\left(\vec{\nabla}\cdot\vec{E}\right)=\mathbf{e}_\mathbf{t}4\pi\rho && \left(\text{time scalar part}\right),\\
&\mathbf{e}_\mathbf{r}\left[\vec{\nabla}\times\vec{H}\right]=i\mathbf{e}_\mathbf{r}\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+i\mathbf{e}_\mathbf{r}\frac{4\pi}{c}\vec{j} && \left(\text{space vector part}\right),\\
&\mathbf{e}_\mathbf{t}\left[\vec{\nabla}\times\vec{E}\right]=-i\mathbf{e}_\mathbf{t}\frac{\partial\vec{H}}{\partial t} && \left(\text{time vector part}\right),\\
&\mathbf{e}_\mathbf{r}\left(\vec{\nabla}\cdot\vec{H}\right)=0 && \left(\text{space scalar part}\right).
\end{aligned} \tag{60}$$

The system (60) coincides with Maxwell equations.

Generalization of Dirac equation

In sedeonic algebra the Dirac equation is written as

$$\left(i\partial_\mathbf{t}-\vec{\nabla}_\mathbf{r}-im_\mathbf{tr}\right)\tilde{\boldsymbol{W}}=0. \tag{61}$$

In fact, this equation describes the special field [21] with zero field intensities $\boldsymbol{E}_0$ and $\vec{\boldsymbol{E}}$ (see expression (45)). In equation (61) the basis elements $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ play the role of the space-time operators, which transform the wave function $\tilde{\boldsymbol{W}}$ by means of component permutation. The matrix forms of field potential $\tilde{\boldsymbol{W}}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ operators are presented in Appendix A.

In special case with the mass equal to zero ehe equation (61) can be rewritten as

$$\left(i\partial_\mathbf{t}-\vec{\nabla}_\mathbf{r}\right)\tilde{\boldsymbol{W}}=0. \tag{62}$$

In fact this equation describes the free massless field of electromagnetic nature (the neutrino field [22]) with field intensities equal to zero (see the expression 57).

Conclusion

Thus in this paper we presented the new version of sixteen-component values “sedeons”, generating associative noncommutative algebra. We proposed the generalized sedeonic second-order wave equation for a massive field and showed that this equation can be represented as the system of first-order Maxwell’s-like equations for the field intensities. The generalized Dirac equation formulated in the sedeonic form was also considered.

We believe that sedeonic algebra is a powerful tool for the analysis of space-time symmetry in relativistic field theory.

Acknowledgments

The authors are very thankful to G.V. Mironova for kind assistance and moral support. We also thank S.A. Korolev for his valuable remarks.

Appendix A: Matrix representation of sedeons

Let us consider a matrix representation of the sedeon. We start with sedeon $\tilde{\boldsymbol{V}}$ in the $\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ basis

$$\tilde{\boldsymbol{V}}=\mathbf{e}_0\bar{\boldsymbol{V}}_0+\mathbf{e}_1\bar{\boldsymbol{V}}_1+\mathbf{e}_2\bar{\boldsymbol{V}}_2+\mathbf{e}_3\bar{\boldsymbol{V}}_3.$$

The sedeonic product of $\mathbf{e_1}$ and $\tilde{\boldsymbol{V}}$ can be written as

$$\mathbf{e_1}\tilde{\boldsymbol{V}} = \mathbf{e_0}\bar{\boldsymbol{V}}_1 + \mathbf{e_1}\bar{\boldsymbol{V}}_0 - i\mathbf{e_2}\bar{\boldsymbol{V}}_3 + i\mathbf{e_3}\bar{\boldsymbol{V}}_2 , \tag{A1}$$

therefore the sedeonic unit $\mathbf{e_1}$ enables the following matrix representation:

$$\mathbf{e_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} . \tag{A2}$$

Analogously:

$$\mathbf{e_0} = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{e_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \ \mathbf{e_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{A3}$$

Using (A2) and (A3) we can write a sedeon $\tilde{\boldsymbol{V}}$ (in $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ basis) in the following matrix form:

$$\tilde{\boldsymbol{V}} = \begin{pmatrix} \bar{\boldsymbol{V}}_0 & \bar{\boldsymbol{V}}_1 & \bar{\boldsymbol{V}}_2 & \bar{\boldsymbol{V}}_3 \\ \bar{\boldsymbol{V}}_1 & \bar{\boldsymbol{V}}_0 & -i\bar{\boldsymbol{V}}_3 & i\bar{\boldsymbol{V}}_2 \\ \bar{\boldsymbol{V}}_2 & i\bar{\boldsymbol{V}}_3 & \bar{\boldsymbol{V}}_0 & -i\bar{\boldsymbol{V}}_1 \\ \bar{\boldsymbol{V}}_3 & -i\bar{\boldsymbol{V}}_2 & i\bar{\boldsymbol{V}}_1 & \bar{\boldsymbol{V}}_0 \end{pmatrix} . \tag{A4}$$

On the other hand we can write sedeon $\tilde{\boldsymbol{V}}$ in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis

$$\tilde{\boldsymbol{V}} = \boldsymbol{V}_0\mathbf{a_0} + \boldsymbol{V}_1\mathbf{a_1} + \boldsymbol{V}_2\mathbf{a_2} + \boldsymbol{V}_3\mathbf{a_3} .$$

Analogously the basis elements $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ have the following representation:

$$\mathbf{a_0} = 1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{a_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \ \mathbf{a_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \ \mathbf{a_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{A5}$$

Using (A5) a sedeon $\tilde{\boldsymbol{V}}$ can be written in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis as a 4×4 matrix

$$\tilde{\boldsymbol{V}} = \begin{pmatrix} \boldsymbol{V}_0 & \boldsymbol{V}_1 & \boldsymbol{V}_2 & \boldsymbol{V}_3 \\ \boldsymbol{V}_1 & \boldsymbol{V}_0 & -i\boldsymbol{V}_3 & i\boldsymbol{V}_2 \\ \boldsymbol{V}_2 & i\boldsymbol{V}_3 & \boldsymbol{V}_0 & -i\boldsymbol{V}_1 \\ \boldsymbol{V}_3 & -i\boldsymbol{V}_2 & i\boldsymbol{V}_1 & \boldsymbol{V}_0 \end{pmatrix} . \tag{A6}$$

Thus the sixteen-component sedeon can be represented as a 16×16 matrix, which can be represented in two different compact 4×4 form. First representation in $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ basis is (A4) with $\bar{\boldsymbol{V}}_n$ components in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis

$$\bar{\boldsymbol{V}}_n = \begin{pmatrix} V_{n0} & V_{n1} & V_{n2} & V_{n3} \\ V_{n1} & V_{n0} & -iV_{n3} & iV_{n2} \\ V_{n2} & iV_{n3} & V_{n0} & -iV_{n1} \\ V_{n3} & -iV_{n2} & iV_{n1} & V_{n0} \end{pmatrix}. \tag{A7}$$

Second representation in $\mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ basis is (A6) with $\boldsymbol{V}_k$ components in $\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ basis

$$\boldsymbol{V}_k = \begin{pmatrix} V_{0k} & V_{1k} & V_{2k} & V_{3k} \\ V_{1k} & V_{0k} & -iV_{3k} & iV_{2k} \\ V_{2k} & iV_{3k} & V_{0k} & -iV_{1k} \\ V_{3k} & -iV_{2k} & iV_{1k} & V_{0k} \end{pmatrix}. \tag{A8}$$

Let us consider the relations between unit vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ and Dirac matrices. Introducing new values

$$\boldsymbol{W}_1 = \frac{1}{2}(\boldsymbol{V}_0 + \boldsymbol{V}_3), \ \boldsymbol{W}_2 = \frac{1}{2}(\boldsymbol{V}_1 + i\boldsymbol{V}_2), \ \boldsymbol{W}_3 = \frac{1}{2}(\boldsymbol{V}_1 - i\boldsymbol{V}_2), \ \boldsymbol{W}_4 = \frac{1}{2}(\boldsymbol{V}_0 - \boldsymbol{V}_3), \tag{A9}$$

we can write the sedeon in the basis of eigenfunctions of operator $\mathbf{a_3}$ in the following form:

$$\tilde{\boldsymbol{V}} = \boldsymbol{W}_1(1+\mathbf{a_3}) + \boldsymbol{W}_2(\mathbf{a_1} - i\mathbf{a_2}) + \boldsymbol{W}_3(\mathbf{a_1} + i\mathbf{a_2}) + \boldsymbol{W}_4(1-\mathbf{a_3}), \tag{A10}$$

where $(1+\mathbf{a_3})$, $(\mathbf{a_1} - i\mathbf{a_2})$, $(\mathbf{a_1} + i\mathbf{a_2})$ and $(1-\mathbf{a_3})$ is a new sedeonic basis. Then the action of vector operators can be represented as

$$\mathbf{a_1}\tilde{\boldsymbol{V}} = \boldsymbol{W}_2(1+\mathbf{a_3}) + \boldsymbol{W}_1(\mathbf{a_1} - i\mathbf{a_2}) + \boldsymbol{W}_4(\mathbf{a_1} + i\mathbf{a_2}) + \boldsymbol{W}_3(1-\mathbf{a_3}),$$

$$\mathbf{a_2}\tilde{\boldsymbol{V}} = -i\boldsymbol{W}_2(1+\mathbf{a_3}) + i\boldsymbol{W}_1(\mathbf{a_1} - i\mathbf{a_2}) - i\boldsymbol{W}_4(\mathbf{a_1} + i\mathbf{a_2}) + i\boldsymbol{W}_3(1-\mathbf{a_3}),$$

$$\mathbf{a_3}\tilde{\boldsymbol{V}} = \boldsymbol{W}_1(1+\mathbf{a_3}) - \boldsymbol{W}_2(\mathbf{a_1} - i\mathbf{a_2}) + \boldsymbol{W}_3(\mathbf{a_1} + i\mathbf{a_2}) - \boldsymbol{W}_4(1-\mathbf{a_3}).$$

Therefore the unit vectors $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ can be written in the new basis as the following matrices:

$$\mathbf{a_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \mathbf{a_2} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \ \mathbf{a_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A11}$$

which coincide with spin operators in Dirac theory [24]:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \ \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

References

1. S.L. Adler – Quaternionic quantum mechanics and quantum fields, New York: Oxford University Press, 1995.
2. V. Majernik – Quaternionic formulation of the classical fields – Advances in Applied Clifford Algebras, **9**(1), 119-130, (1999).

3. K. Imaeda – A new formulation of classical electrodynamics - Nuovo cimento, **32**(1), 138-162, (1976).
4. A.J. Davies – Quaternionic Dirac equation, Physical Review D, **41**(8), 2628-2630 (1990).
5. C. Schwartz – Relativistic quaternionic wave equation, Journal of Mathematical Physics, **47**, 122301 (2006).
6. L.Yu-Fen – Triality, biquaternion and vector representation of the Dirac equation, Advances in Applied Clifford Algebras, **12**(2), 109-124 (2002).
7. M. Gogberashvili – Octonionic electrodynamics – Journal of Physics A: Mathematics in General, **39**, 7099-7104, (2006).
8. A. Gamba – Maxwell's equations in octonion form - Nuovo Cimento A, **111**(3), 293-302, (1998).
9. T. Tolan, K. Özdas, M. Tanisli – Reformulation of electromagnetism with octonions - Nuovo Cimento B, **121**(1), 43-55, (2006).
10. M. Gogberashvili - Octonionic version of Dirac equations, International Journal of Modern Physics A, **21**(17), 3513-3523 (2006).
11. S. De Leo, K. Abdel-Khalek – Octonionic Dirac equation, Progress in Theoretical Physics, **96**, 833-846 (1996).
12. D. Hestenes – Observables, operators, and complex numbers in the Dirac theory, Journal of Mathematical Physisc, **16**, 556-572 (1975).
13. K. Carmody – Circular and hyperbolic quaternions, octonions, and sedenions, Applied Mathematics and Computations, **28**, 47-72 (1988).
14. K. Carmody – Circular and hyperbolic quaternions, octonions, and sedenions – further results, Appl. Math. Comput., **84**, 27-47 (1997).
15. K. Imaeda, M. Imaeda – Sedenions: algebra and analysis, Appl. Math. Comp., **115**, 77-88 (2000).
16. J. Köplinger – Dirac equation on hyperbolic octonions, Appl. Math. Comput., **182**, 443-446 (2006).
17. W.P. Joyce – Dirac theory in spacetime algebra: I. The generalized bivector Dirac equation, J. Phys. A: Math. Gen., **34**, 1991-2005 (2001).
18. C. Cafaro, S.A. Ali – The spacetime algebra approach to massive classical electrodynamics with magnetic monopoles, Advances in Applied Clifford Algebras, **17**, 23-36 (2006).
19. V.L. Mironov, S.V. Mironov – Octonic representation of electromagnetic field equations, J. Math. Phys., **50**, 012901 (2009).
20. V.L. Mironov, S.V. Mironov – Octonic second-order equations of relativistic quantum mechanics, J. Math. Phys., **50**, 012302 (2009).
21. V.L. Mironov, S.V.Mironov – Octonic first-order equations of relativistic quantum mechanics, International Journal of Modern Physics A, **24**(22), 4157-4167 (2009).
22. V.L. Mironov, S.V.Mironov – Sedeonic generalization of relativistic quantum mechanics, International Journal of Modern Physics A, **24**(32), 6237-6254 (2009).
23. L.D. Landau and E.M.Lifshits – Classical Theory of Fields (4th ed.). New York: Pergamon Press, Vol. 2 of the Course of Theoretical Physics. (1975).
24. P.A.M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, Oxford, 1958).