Sedeonic theory of massive fields

Sergey V. Mironov and Victor L. Mironov*

Institute for physics of microstructures RAS, Nizhniy Novgorod, Russia

(Submitted on August 5, 2013)

Abstract

In the present paper we develop the description of massive fields on the basis of space-time algebra of sixteen-component sedeons. The generalized sedeonic second-order equation for the potential of baryon field is proposed. It is shown that this equation can be reformulated in the form of a system of Maxwell-like equations for the field intensities. We calculate the baryonic fields in the simple model of point baryon charge and obtain the expression for the baryon-baryon interaction energy. We also propose the generalized sedeonic first-order equation for the potential of lepton field and calculate the energies of lepton-lepton and lepton-baryon interactions.

Introduction

Historically, the first theory of nuclear forces based on hypothesis of one-meson exchange has been formulated by Hideki Yukawa in 1935 [1]. In fact, he postulated a nonhomogeneous equation for the scalar field similar to the Klein-Gordon equation, whose solution is an effective short-range potential (so-called Yukawa potential). This theory clarified the short-ranged character of nuclear forces and was successfully applied to the explaining of low-energy nucleon-nucleon scattering data and properties of deuteron [2]. Afterwards during 1950s - 1990s many-meson exchange theories and phenomenological approach based on the effective nucleon potentials were proposed for the description of few-nucleon systems and intermediate-range nucleon interactions [3-11]. Nowadays the main fundamental concepts of nuclear force theory are developed in the frame of quantum chromodynamics (so-called effective field theory [13-15], see also reviews [16,17]) however, the meson theories and the effective potential approaches still play an important role for experimental data analysis in nuclear physics [18, 19].

The attempts to generalize the Klein-Gordon equation on the basis of the Clifford algebras and different systems of hypercomplex numbers have been made in [20-24]. The authors of these papers discussed the possibility of constructing the field equations similar to the equations of electrodynamics but with a massive "photon". The resulting Proca-Maxwell equations enclose field's intensities and potentials. However, as we have shown in [25] octonic Klein-Gordon equation can be reformulated as a system of first-order equations, which contain only field's strengths.

On the other hand, there are a lot of studies concerning the generalization of the Dirac equation on the basis of different systems of hypercomplex numbers [26-30], but the interpretation of this equation as the equation for a special potential field with a zero field strength zero was given in [25].

In the present paper we develop a description of massive fields based on analogies with the similar description of massless fields in electrodynamics. For this purpose we use the sedeonic formalism [31-33] developed on the basis of octonic algebra introduced previously in [34,35]. We consider the baryon field, which is described by sedeonic second-order equation, as well as the lepton field that is described by first-order equation. The expressions for the energies of baryon-baryon, lepton-lepton and lepton-baryon interactions are calculated and analyzed.

*E-mail: mironov@ipmras.ru

1 Sedeonic space-time algebra

To begin with we briefly review the basic properties of sedeons. The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

1. Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
2. Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
3. Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
4. Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}}. \tag{1}$$

Let us introduce scalar-vector basis $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, where the value $\mathbf{a_0} \equiv 1$ is absolute scalar unit and the values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are absolute unit vectors generating the right Cartesian basis. We introduce also four space-time scalar units $\mathbf{e_0}$, $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, where value $\mathbf{e_0} \equiv 1$ is a absolute scalar unit; $\mathbf{e_1} \equiv \mathbf{e_t}$ is a time scalar unit; $\mathbf{e_2} \equiv \mathbf{e_r}$ is a space scalar unit; $\mathbf{e_3} \equiv \mathbf{e_{tr}}$ is a space-time scalar unit. Using space-time scalar units $\mathbf{e_j}$ ($\mathbf{j} = 0, 1, 2, 3$) and scalar-vector basis $\mathbf{a_k}$ ($\mathbf{k} = 0, 1, 2, 3$) we can introduce unified sedeonic components V_{jk} in accordance with the following relations:

$$\begin{aligned}
&V = \mathbf{e_0} V_{00} \mathbf{a_0}, \\
&\vec{V} = \mathbf{e_0} \left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right), \\
&V_{\mathbf{t}} = \mathbf{e_1} V_{10} \mathbf{a_0}, \\
&\vec{V}_{\mathbf{t}} = \mathbf{e_1} \left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right), \\
&V_{\mathbf{r}} = \mathbf{e_2} V_{20} \mathbf{a_0}, \\
&\vec{V}_{\mathbf{r}} = \mathbf{e_2} \left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right), \\
&V_{\mathbf{tr}} = \mathbf{e_3} V_{30} \mathbf{a_0}, \\
&\vec{V}_{\mathbf{tr}} = \mathbf{e_3} \left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{2}$$

Then the sedeon (1) can be written in the following expanded form:

$$\begin{aligned}
\tilde{\mathbf{V}} = \; &\mathbf{e_0} \left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) \\
&+\mathbf{e_1} \left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) \\
&+\mathbf{e_2} \left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right) \\
&+\mathbf{e_3} \left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{3}$$

The sedeonic components V_{jk} are numbers (complex in general). Further we will use symbol 1 instead of units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and space-time units $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are presented in tables 1 and 2 respectively.

In the tables and further the value i is the imaginary unit ($i^2 = -1$). Note that sedeonic units $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ and unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ generate the anticommutative algebras:

$$\begin{aligned}
\mathbf{a_n}\mathbf{a_m} &= -\mathbf{a_m}\mathbf{a_n}, \\
\mathbf{e_n}\mathbf{e_m} &= -\mathbf{e_m}\mathbf{e_n},
\end{aligned} \tag{4}$$

for $\mathbf{n}$ and $\mathbf{m} = 1, 2, 3$ ($\mathbf{n} \neq \mathbf{m}$), but $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ commute with $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$:

$$\mathbf{a_n}\mathbf{e_m} = \mathbf{e_m}\mathbf{a_n}, \tag{5}$$

Table 1:

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2:

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$
$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$
$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1

for any $\mathbf{n}$ and $\mathbf{m}$.

Thus the sedeon $\tilde{\mathbf{V}}$ is the complicated space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

Introducing the designations of scalar-vector values

$$\begin{aligned}
\overline{\mathbf{V}}_0 &= V_{00} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3},\\
\overline{\mathbf{V}}_1 &= V_{10} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3},\\
\overline{\mathbf{V}}_2 &= V_{20} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3},\\
\overline{\mathbf{V}}_3 &= V_{30} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3},
\end{aligned} \tag{6}$$

we can write the sedeon (3) in the compact form

$$\tilde{\mathbf{V}} = \overline{\mathbf{V}}_0 + \mathbf{e_1}\overline{\mathbf{V}}_1 + \mathbf{e_2}\overline{\mathbf{V}}_2 + \mathbf{e_3}\overline{\mathbf{V}}_3. \tag{7}$$

On the other hand, introducing the designations of space-time sedeon-scalars

$$\begin{aligned}
\mathbf{V}_0 &= V_{00} + \mathbf{e_1}V_{10} + \mathbf{e_2}V_{20} + \mathbf{e_3}V_{30},\\
\mathbf{V}_1 &= V_{01} + \mathbf{e_1}V_{11} + \mathbf{e_2}V_{21} + \mathbf{e_3}V_{31},\\
\mathbf{V}_2 &= V_{02} + \mathbf{e_1}V_{12} + \mathbf{e_2}V_{22} + \mathbf{e_3}V_{32},\\
\mathbf{V}_3 &= V_{03} + \mathbf{e_1}V_{13} + \mathbf{e_2}V_{23} + \mathbf{e_3}V_{33},
\end{aligned} \tag{8}$$

we can write the sedeon (3) in another form

$$\tilde{\mathbf{V}} = \mathbf{V}_0 + \mathbf{V}_1\mathbf{a_1} + \mathbf{V}_2\mathbf{a_2} + \mathbf{V}_3\mathbf{a_3}, \tag{9}$$

or introducing the sedeon-vector

$$\vec{\mathbf{V}} = \vec{V} + \vec{V}_{\mathbf{t}} + \vec{V}_{\mathbf{r}} + \vec{V}_{\mathbf{tr}} = \mathbf{V}_1\mathbf{a_1} + \mathbf{V}_2\mathbf{a_2} + \mathbf{V}_3\mathbf{a_3}, \tag{10}$$

it can be represented in following compact form:

$$\tilde{\mathbf{V}} = \mathbf{V}_0 + \vec{\mathbf{V}}. \tag{11}$$

Further we will indicate the sedeon-scalars and the sedeon-vectors with the bold capital letters.

Let us consider the sedeonic multiplication in detail. The sedeonic product of two sedeons $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ can be presented in the following form:

$$\begin{aligned}
\tilde{\mathbf{A}}\tilde{\mathbf{B}} &= \left(\mathbf{A}_0 + \vec{\mathbf{A}}\right)\left(\mathbf{B}_0 + \vec{\mathbf{B}}\right)\\
&= \mathbf{A}_0\mathbf{B}_0 + \mathbf{A}_0\vec{\mathbf{B}} + \vec{\mathbf{A}}\mathbf{B}_0 + \left(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}\right) + \left[\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right].
\end{aligned} \tag{12}$$

Here we denote the sedeonic scalar multiplication of two sedeon-vectors (internal product) by symbol "·" and round brackets

$$\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3, \tag{13}$$

and sedeonic vector multiplication (external product) by symbol "×" and square brackets

$$\begin{aligned}\left[\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right] &= i\left(\mathbf{A}_2\mathbf{B}_3 - \mathbf{A}_3\mathbf{B}_2\right)\mathbf{a_1} + i\left(\mathbf{A}_3\mathbf{B}_1 - \mathbf{A}_1\mathbf{B}_3\right)\mathbf{a_2} \\ &+ i\left(\mathbf{A}_1\mathbf{B}_2 - \mathbf{A}_2\mathbf{B}_1\right)\mathbf{a_3}.\end{aligned} \tag{14}$$

In (13) and (14) the multiplication of sedeonic components is performed in accordance with (8) and table 2. Note that in sedeonic algebra the expression for the vector product has some difference from analogous expression in Gibbs vector algebra. Let us consider three absolute vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Then the formula for the vector triple product in sedeonic algebra has the following form:

$$\left[\vec{A} \times \left[\vec{B} \times \vec{C}\right]\right] = -\vec{B}\left(\vec{A} \cdot \vec{C}\right) + \vec{C}\left(\vec{A} \cdot \vec{B}\right). \tag{15}$$

Thus, the sedeonic product

$$\tilde{\mathbf{F}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} = \mathbf{F}_0 + \vec{\mathbf{F}} \tag{16}$$

has the following components:

$$\begin{aligned}\mathbf{F}_0 &= \mathbf{A}_0\mathbf{B}_0 + \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3, \\ \mathbf{F}_1 &= \mathbf{A}_1\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_1 + i\mathbf{A}_2\mathbf{B}_3 - i\mathbf{A}_3\mathbf{B}_2, \\ \mathbf{F}_2 &= \mathbf{A}_2\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_2 + i\mathbf{A}_3\mathbf{B}_1 - i\mathbf{A}_1\mathbf{B}_3, \\ \mathbf{F}_3 &= \mathbf{A}_3\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_3 + i\mathbf{A}_1\mathbf{B}_2 - i\mathbf{A}_2\mathbf{B}_1.\end{aligned} \tag{17}$$

2 Baryon field

The hypothesis that free baryon field is described by the second order wave equation on Clifford algebra was formulated recently in Ref. 36. Here we develop this hypothesis on the basis of sedeonic potentials.

2.1 Generalized sedeonic wave equation for massive baryon field

Let us consider the field potential in the form of space-time sedeon

$$\tilde{\mathbf{W}}(\vec{r}, t) = \mathbf{W}_0(\vec{r}, t) + \vec{\mathbf{W}}(\vec{r}, t). \tag{18}$$

The potential of free massive field should satisfy an equation, which is obtained from the Einstein relation between field's energy and momentum

$$E^2 - c^2p^2 = m_0^2c^4 \tag{19}$$

by means of changing classical energy E and momentum $\vec{p}$ on corresponding quantum-mechanical operators:

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad \hat{\vec{p}} = -i\hbar\vec{\nabla}, \tag{20}$$

where the absolute gradient vector has the following form:

$$\vec{\nabla} = \frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3. \tag{21}$$

Here c is the velocity of light, $\hbar$ is the Planck constant and the parameter m_0 can be interpreted as the mass of a quantum of baryonic field (or in accordance with generally accepted concept as the mass of a meson field quantum). Using sedeonic algebra the Einstein's relation (19) can be represented in the following form:

$$(i\mathbf{e_1}E + \mathbf{e_2}c\vec{p} + \mathbf{e_3}m_0c^2)(i\mathbf{e_1}E + \mathbf{e_2}c\vec{p} + \mathbf{e_3}m_0c^2) = 0. \tag{22}$$

Then the generalized sedeonic wave equation for free massive field can be written in the symmetric form

$$\left(i\mathbf{e_1}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}\frac{m_0c}{\hbar}\right)\left(i\mathbf{e_1}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}\frac{m_0c}{\hbar}\right)\tilde{\mathbf{W}} = 0. \tag{23}$$

For convenience we introduce new operators

$$\begin{aligned} \partial &= \frac{1}{c}\frac{\partial}{\partial t}, \\ m &= \frac{m_0c}{\hbar}. \end{aligned} \tag{24}$$

Then we can rewrite the equation (23) in compact form:

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\tilde{\mathbf{W}} = 0. \tag{25}$$

Let us choose the potential in the following form:

$$\tilde{\mathbf{W}} = a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}, \tag{26}$$

where the components $a, b, c, d, \vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ are the functions of spatial coordinates and time. Introducing the scalar and vector fields strengths according to the following definitions:

$$\begin{aligned} e &= \partial b + (\vec{\nabla}\cdot\vec{C}) + md, \\ f &= \partial a + (\vec{\nabla}\cdot\vec{D}) + mc, \\ g &= \partial d + (\vec{\nabla}\cdot\vec{A}) - mb, \\ h &= \partial c + (\vec{\nabla}\cdot\vec{B}) - ma, \\ \vec{E} &= -\partial\vec{B} - \vec{\nabla}c - i[\vec{\nabla}\times\vec{C}] - m\vec{D}, \\ \vec{F} &= -\partial\vec{A} - \vec{\nabla}d + i[\vec{\nabla}\times\vec{D}] - m\vec{C}, \\ \vec{G} &= -\partial\vec{D} - \vec{\nabla}a - i[\vec{\nabla}\times\vec{A}] + m\vec{B}, \\ \vec{H} &= -\partial\vec{C} - \vec{\nabla}b + i[\vec{\nabla}\times\vec{B}] + m\vec{A}, \end{aligned} \tag{27}$$

we get

$$\begin{aligned} &\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right) \\ &= -e + i\mathbf{e_1}f - i\mathbf{e_2}g + i\mathbf{e_3}h - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_2}\vec{G} + \mathbf{e_3}\vec{H} \end{aligned} \tag{28}$$

and the wave equation (25) takes the form

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\left(-e + i\mathbf{e_1}f - i\mathbf{e_2}g + i\mathbf{e_3}h - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_2}\vec{G} + \mathbf{e_3}\vec{H}\right) = 0. \tag{29}$$

Performing the action of operator in the left part of the equation (29), and separating the terms with different space-time properties, we obtain the system of equations for the field's strengths, similar to the system of Maxwell's equations in electrodynamics:

$$
\begin{aligned}
&\partial f + (\vec{\nabla} \cdot \vec{G}) - mh = 0, \\
&\partial e + (\vec{\nabla} \cdot \vec{H}) - mg = 0, \\
&\partial h + (\vec{\nabla} \cdot \vec{E}) + mf = 0, \\
&\partial g + (\vec{\nabla} \cdot \vec{F}) + me = 0, \\
&\partial \vec{F} + \vec{\nabla} g + i[\vec{\nabla} \times \vec{G}] - m\vec{H} = 0, \\
&\partial \vec{E} + \vec{\nabla} h - i[\vec{\nabla} \times \vec{H}] - m\vec{G} = 0, \\
&\partial \vec{H} + \vec{\nabla} e + i[\vec{\nabla} \times \vec{E}] + m\vec{F} = 0, \\
&\partial \vec{G} + \vec{\nabla} f - i[\vec{\nabla} \times \vec{F}] + m\vec{E} = 0.
\end{aligned}
\tag{30}
$$

Of course, in the case of zero-mass the scalar fields can be chosen zero due to the Lorentz gauge, and then the equations (30) can be reduced to the well-known Maxwell's equations. Multiplying each of the equations (30) to the corresponding field strength and adding these equations to each other, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\partial \left(f^2 + e^2 + h^2 + g^2 + \vec{F}^2 + \vec{E}^2 + \vec{H}^2 + \vec{G}^2 \right) \\
&+ f\left(\vec{\nabla} \cdot \vec{G}\right) + e\left(\vec{\nabla} \cdot \vec{H}\right) + h\left(\vec{\nabla} \cdot \vec{E}\right) + g\left(\vec{\nabla} \cdot \vec{F}\right) \\
&+ \left(\vec{F} \cdot \vec{\nabla} g\right) + \left(\vec{E} \cdot \vec{\nabla} h\right) + \left(\vec{H} \cdot \vec{\nabla} e\right) + \left(\vec{G} \cdot \vec{\nabla} f\right) \\
&+ i\left(\vec{F} \cdot [\vec{\nabla} \times \vec{G}]\right) - i\left(\vec{E} \cdot [\vec{\nabla} \times \vec{H}]\right) \\
&+ i\left(\vec{H} \cdot [\vec{\nabla} \times \vec{E}]\right) - i\left(\vec{G} \cdot [\vec{\nabla} \times \vec{F}]\right) = 0.
\end{aligned}
\tag{31}
$$

Let us introduce the following notations:

$$
w = -\frac{1}{8\pi}\left(f^2 + e^2 + h^2 + g^2 + \vec{F}^2 + \vec{E}^2 + \vec{H}^2 + \vec{G}^2 \right), \tag{32}
$$

$$
\vec{P} = -\frac{c}{4\pi}\left(e\vec{H} + f\vec{G} + g\vec{F} + h\vec{E} + i\left[\vec{E} \times \vec{H}\right] + i\left[\vec{G} \times \vec{F}\right] \right). \tag{33}
$$

Then the equation (31) can be written as:

$$
\frac{1}{c}\frac{\partial w}{\partial t} + \left(\vec{\nabla} \cdot \vec{P}\right) = 0. \tag{34}
$$

This is an analog of the Poynting theorem for massive fields. The value w plays the role of the field energy density and $\vec{P}$ is a vector of energy flux density. The minus sign in expressions (32) and (33) are chosen with respect to the attractive character of baryon-baryon interaction.

2.2 Nonhomogeneous equation for baryon field

Here we consider sedeonic equations for baryon field with phenomenological baryon source. We assume that the potential of the baryonic field is described by sedeonic nonhomogeneous wave equation for massive field:

$$
\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m \right)\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m \right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}}. \tag{35}
$$

By analogy with electrodynamics [32] we consider the source of baryon field in the form of incomplete sedeon

$$
\tilde{\mathbf{J}} = i\mathbf{e_1}4\pi\rho_B + \mathbf{e_2}\frac{4\pi}{c}\vec{j}_B, \tag{36}
$$

where ρ_B is a volume density of baryon charge and $\vec{j}_B$ is density of baryon current. In this case we can describe the baryon field by sedeonic potential $\tilde{\mathbf{W}}$ written in the following form

$$\tilde{\mathbf{W}} = i\mathbf{e_1}b + \mathbf{e_2}\vec{C}, \tag{37}$$

where $b(\vec{r},t)$ is a scalar part and $\vec{C}(\vec{r},t)$ is a vector part of baryon potential. In this case we have only the following nonzero field's strengths

$$\begin{aligned}
&e = \partial b + \left(\vec{\nabla} \cdot \vec{C}\right), \\
&g = -mb, \\
&\vec{E} = -i\left[\vec{\nabla} \times \vec{C}\right], \\
&\vec{F} = -m\vec{C}, \\
&\vec{H} = -\partial\vec{C} - \vec{\nabla}b.
\end{aligned} \tag{38}$$

Then we obtain the following equations for the field strengths:

$$\begin{aligned}
&\partial e + (\vec{\nabla} \cdot \vec{H}) - mg = -4\pi\rho_B, \\
&(\vec{\nabla} \cdot \vec{E}) = 0, \\
&\partial g + (\vec{\nabla} \cdot \vec{F}) + me = 0, \\
&\partial\vec{F} + \vec{\nabla}g - m\vec{H} = 0, \\
&\partial\vec{E} - i[\vec{\nabla} \times \vec{H}] = 0, \\
&\partial\vec{H} + \vec{\nabla}e + i[\vec{\nabla} \times \vec{E}] + m\vec{F} = \frac{4\pi}{c}\vec{j}_B, \\
&i[\vec{\nabla} \times \vec{F}] - m\vec{E} = 0.
\end{aligned} \tag{39}$$

2.3 Stationary field of point scalar source

In the stationary case $\vec{j}_B = 0$ and potential of the field can be chosen as a scalar potential

$$\tilde{\mathbf{W}} = i\mathbf{e_1}b\left(\vec{r}\right). \tag{40}$$

Then we have only two nonzero field components

$$\begin{aligned}
&g = -mb, \\
&\vec{H} = -\vec{\nabla}b
\end{aligned} \tag{41}$$

and the following field equations:

$$\begin{aligned}
&(\vec{\nabla} \cdot \vec{H}) - mg = -4\pi\rho_B, \\
&\vec{\nabla}g - m\vec{H} = 0, \\
&[\vec{\nabla} \times \vec{H}] = 0.
\end{aligned} \tag{42}$$

Let us calculate the baryon field produced by a scalar stationary point baryon source

$$\tilde{\mathbf{J}} = -4\pi q_B\delta(\vec{r}), \tag{43}$$

where q_B is the point baryon charge and $\delta(\vec{r})$ is delta function. Then stationary wave equation can be written in spherical coordinates as

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{m_0^2c^2}{\hbar^2}\right)b(r) = -4\pi q_B\delta(\vec{r}). \tag{44}$$

The partial solution of the equation (44), which decays at $r \to \infty$, is

$$b = \frac{q_B}{r} \exp\left(-\frac{m_0 c}{\hbar} r\right). \tag{45}$$

Thus the stationary baryon field has scalar and vector components

$$g = \frac{m_0 c}{\hbar} \frac{q_B}{r} \exp\left(-\frac{m_0 c}{\hbar} r\right), \tag{46}$$

$$\vec{H} = \left(\frac{1}{r} + \frac{m_0 c}{\hbar}\right) \frac{q_B}{r} \exp\left(-\frac{m_0 c}{\hbar} r\right) \vec{r}_0, \tag{47}$$

where $\vec{r}_0$ is a unit radial vector.

2.4 Baryon-baryon interaction

Let us consider the interaction of two point baryons due to the overlap of their fields. Taking into account that the baryon field in this case is the sum of the two fields $g = g_1 + g_2$ and $\vec{H} = \vec{H}_1 + \vec{H}_2$ the energy of interaction is equal [see (32)]

$$W_{BB} = -\frac{1}{8\pi} \int \{g_1 g_2 + (H_1 \cdot H_2)\} dV, \tag{48}$$

where the integral is over all space. This expression can be derived analytically:

$$W_{BB} = -\frac{q_{B1} q_{B2}}{R} \exp\left(-\frac{m_0 c}{\hbar} R\right), \tag{49}$$

where R is the distance between the baryons.

3 Lepton field

The hypothesis that free lepton field is described by the first-order Dirac-like wave equation on Clifford algebra was formulated recently in Ref. 36. Here we develop this hypothesis on the basis of sedeonic potentials.

3.1 Homogeneous sedeonic equation for lepton field

In sedeonic algebra the homogeneous first-order Dirac-like equation is written as

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right) \tilde{\mathbf{W}} = 0. \tag{50}$$

In equation (50) the basis elements $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ and $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ play the role of the space-time operators, which transform the wave function by means of component permutation. Choosing potential in the form (26) we find that sedeonic equation (50) is equivalent to the following system

$$\begin{aligned}
&\partial a + (\vec{\nabla} \cdot \vec{D}) + mc = 0, \\
&\partial b + (\vec{\nabla} \cdot \vec{C}) + md = 0, \\
&\partial c + (\vec{\nabla} \cdot \vec{B}) - ma = 0, \\
&\partial d + (\vec{\nabla} \cdot \vec{A}) - mb = 0, \\
&\partial \vec{A} + \vec{\nabla} d - i[\vec{\nabla} \times \vec{D}] + m\vec{C} = 0, \\
&\partial \vec{B} + \vec{\nabla} c + i[\vec{\nabla} \times \vec{C}] + m\vec{D} = 0, \\
&\partial \vec{C} + \vec{\nabla} b - i[\vec{\nabla} \times \vec{B}] - m\vec{A} = 0, \\
&\partial \vec{D} + \vec{\nabla} a + i[\vec{\nabla} \times \vec{A}] - m\vec{B} = 0.
\end{aligned} \tag{51}$$

In fact, these equations describe the special field [25] with zero field intensities (see for comparison the expressions (27)).

3.2 Plane wave solution

Let us consider the plane wave solution of equation (50) in detail. In this case the potential can be written as

$$\tilde{\mathbf{W}} = \tilde{\mathbf{U}}\ \exp\left\{-i\omega t + i\left(\vec{k}\cdot\vec{r}\right)\right\}, \tag{52}$$

where ω is a frequency and $\vec{k}$ is an absolute wave vector; the amplitude of the wave $\mathbf{U}$ does not depend on the coordinates and time. In this case, the dependence of frequency on the wave vector has two branches:

$$\omega_{\pm} = \pm\sqrt{c^2k^2 + \frac{m_0^2c^4}{\hbar^2}}\ . \tag{53}$$

Let us consider the amplitude of the wave function in the form of (26):

$$\tilde{\mathbf{U}} = a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}, \tag{54}$$

where a, b, c, d, $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ are arbitrary constants. Then the solution can be written as

$$\tilde{\mathbf{W}} = \left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right)\exp\left\{-i\omega_{\pm}t + i\left(\vec{k}\cdot\vec{r}\right)\right\}. \tag{55}$$

Substituting this expression in the original equation (50) we get:

$$\left(\mathbf{e_1}\frac{\omega_{\pm}}{c} - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}\frac{m_0c}{\hbar}\right)\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right) = 0. \tag{56}$$

For convenience we introduce the following notation:

$$\begin{aligned} \omega' &= \frac{\omega_{\pm}}{c}, \\ m &= \frac{m_0c}{\hbar} \end{aligned} \tag{57}$$

and equation (56) can be rewritten as

$$\left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right) = 0. \tag{58}$$

For fixed $\vec{k}$ let us represent the vector constants in (54) in the form

$$\begin{aligned} \vec{A} &= \vec{A}_{\|} + \vec{A}_{\perp}, \\ \vec{B} &= \vec{B}_{\|} + \vec{B}_{\perp}, \\ \vec{C} &= \vec{C}_{\|} + \vec{C}_{\perp}, \\ \vec{D} &= \vec{D}_{\|} + \vec{D}_{\perp}, \end{aligned} \tag{59}$$

where the vectors $\vec{A}_{\|}$, $\vec{B}_{\|}$, $\vec{C}_{\|}$ and $\vec{D}_{\|}$ are parallel to the vector $\vec{k}$ while the vectors $\vec{A}_{\perp}$, $\vec{B}_{\perp}$, $\vec{C}_{\perp}$ and $\vec{D}_{\perp}$ are perpendicular to $\vec{k}$. Then performing the multiplication in (58), we obtain the following system of algebraic equations:

$$i\omega' b - ikC_{\|} - md = 0, \tag{60}$$

$$\omega' a - kD_{\|} + imc = 0, \tag{61}$$

$$-\omega' d + kA_{\|} + imb = 0, \tag{62}$$

$$\omega' c - kB_{\|} - ima = 0, \tag{63}$$

$$\omega' B_{\|} - kc + imD_{\|} = 0, \tag{64}$$

$$i\omega' A_{\|} - ikd - mC_{\|} = 0, \tag{65}$$

$$i\omega' D_\parallel - ika + mB_\parallel = 0, \tag{66}$$
$$i\omega' C_\parallel - ikb + mA_\parallel = 0, \tag{67}$$
$$\omega' \vec{B}_\perp - i\left[\vec{k}\times\vec{C}_\perp\right] + im\vec{D}_\perp = 0, \tag{68}$$
$$i\omega' \vec{A}_\perp - \left[\vec{k}\times\vec{D}_\perp\right] - m\vec{C}_\perp = 0, \tag{69}$$
$$i\omega' \vec{D}_\perp + \left[\vec{k}\times\vec{A}_\perp\right] + m\vec{B}_\perp = 0, \tag{70}$$
$$i\omega' \vec{C}_\perp - \left[\vec{k}\times\vec{B}_\perp\right] + m\vec{A}_\perp = 0, \tag{71}$$

where the values $A_\parallel$, $B_\parallel$, $C_\parallel$ and $D_\parallel$ are the projections of the vectors $\vec{A}_\parallel$, $\vec{B}_\parallel$, $\vec{C}_\parallel$ and $\vec{D}_\parallel$ on the vector $\vec{k}$.

Let us solve this system of equations. From (70) and (71) we find

$$\vec{D}_\perp = \frac{im}{\omega'}\vec{B}_\perp + \frac{i}{\omega'}\left[\vec{k}\times\vec{A}_\perp\right], \tag{72}$$
$$\vec{C}_\perp = \frac{im}{\omega'}\vec{A}_\perp - \frac{i}{\omega'}\left[\vec{k}\times\vec{B}_\perp\right]. \tag{73}$$

Using (53) one can easily check that for arbitrary vector constants $A_\perp$ and $B_\perp$ equations (68) and (69) are fulfilled.

As a next step from equations (60)-(63) we obtain:

$$C_\parallel = \frac{\omega'}{k}b + i\frac{m}{k}d, \tag{74}$$
$$D_\parallel = \frac{\omega'}{k}a + i\frac{m}{k}c, \tag{75}$$
$$A_\parallel = \frac{\omega'}{k}d - i\frac{m}{k}b, \tag{76}$$
$$B_\parallel = \frac{\omega'}{k}c - i\frac{m}{k}a. \tag{77}$$

One can check that these solution fulfill the equations (64)-(67).

Thus the sedeon $\tilde{\mathbf{U}}$ has the form

$$\begin{aligned}\tilde{\mathbf{U}} &= a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d \\ &+ \{i\omega' d + mb + \mathbf{e_1}\omega' c - i\mathbf{e_1}ma + \mathbf{e_2}\omega' b + i\mathbf{e_2}md - \mathbf{e_3}\omega' a - i\mathbf{e_3}mc\}\frac{\vec{k}}{k^2} \\ &+ i\vec{A}_\perp + \mathbf{e_1}\vec{B}_\perp + i\mathbf{e_2}\frac{m}{\omega'}\vec{A}_\perp - i\mathbf{e_3}\frac{m}{\omega'}\vec{B}_\perp - i\mathbf{e_3}\frac{1}{\omega'}\left[\vec{k}\times\vec{A}_\perp\right] - i\mathbf{e_2}\frac{1}{\omega'}\left[\vec{k}\times\vec{B}_\perp\right].\end{aligned} \tag{78}$$

Note that the expression (78) can be rewritten in the following form:

$$\tilde{\mathbf{U}} = \left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\left\{i\mathbf{e_2}\frac{\vec{k}}{k^2}\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d\right) + i\mathbf{e_1}\frac{1}{\omega'}\vec{A}_\perp + \frac{1}{\omega'}\vec{B}_\perp\right\}. \tag{79}$$

Substituted this amplitude into (58) one can see that this equation is satisfied for any parameters $a, b, c, d, \vec{A}_\perp, \vec{B}_\perp$ because the expression in round brackets is sedeonic zero divisor. Indeed it is simple to check that

$$\left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right) = 0. \tag{80}$$

It is worth mention that the expression (79) is defined by 8 scalar constants. Nevertheless one can represent the general form of the plane-wave solution as

$$\tilde{\mathbf{W}} = \left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\tilde{\mathbf{M}}\exp\left\{-i\omega t + i\left(\vec{k}\cdot\vec{r}\right)\right\}, \tag{81}$$

where $\tilde{\mathbf{M}}$ is an arbitrary sedeon defined by 16 scalar constants. In this case after performing multiplication in (81) one will obtain that the components of the resulting sedeon will be defined only by 8 independent combinations of the sedeon $\tilde{\mathbf{M}}$ components.

3.3 Nonhomogeneous equation for lepton field

Let us consider the nonhomogeneous equation corresponding to the equation (50)

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\tilde{\mathbf{W}} = \tilde{\mathbf{I}}. \tag{82}$$

Here the sedeonic source $\tilde{\mathbf{I}}$ describes the lepton charges and the corresponding currents. Choosing the potential in the form (26), we obtain the following equation for the lepton fields:

$$-e + i\mathbf{e_1}f - i\mathbf{e_2}g + i\mathbf{e_3}h - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_2}\vec{G} + \mathbf{e_3}\vec{H} = \mathbf{I}_0 + \vec{\mathbf{I}}. \tag{83}$$

This equation means that the intensity of the lepton fields are nonzero only in the region of field source.

Let us consider the a stationary lepton field generated by a scalar point source

$$\mathbf{I}_0 = -i\mathbf{e_2}q_L\delta(\vec{r}), \tag{84}$$

where q_L is the point lepton charge. Then the intensity of the scalar lepton field is

$$g_L(\vec{r}) = q_L\delta(\vec{r}). \tag{85}$$

Since this field is non-zero only in the region of source, it indicates that two point leptons interact only if they are in the same point in space. The interaction energy for two point leptons is equal

$$W_{LL} = -\frac{1}{8\pi}\int g_{L1}g_{L2}dV = -q_{L1}q_{L2}\delta(\vec{R}), \tag{86}$$

where $\vec{R}$ is the vector of distance between leptons.

4 Baryon - lepton interaction

The interaction of baryon and lepton is due to the overlap of scalar fields g_B and g_L. In the case of a point baryon and lepton the fields are determined by the expressions (46) and (85), so that the interaction energy is equal to:

$$W_{BL} = -\frac{1}{8\pi}\int g_B g_L dV. \tag{87}$$

As a result, we get:

$$W_{BL} = -\frac{m_0 c}{\hbar}\frac{g_B g_L}{R}\exp\left(-\frac{m_0 c}{\hbar}R\right), \tag{88}$$

where R is the distance between the baryon and lepton. In particular, the baryon-lepton interaction can lead to the formation of a stable proton-electron bound state, which can be interpreted as a neutron.

5 Summary

Thus, in this paper we considered the sedeonic generalization of equations describing the massive fields. It was shown that this approach allows to construct the theory of massive fields analogous to the theory of massless electromagnetic field in classical electrodynamics.

We considered the sedeonic second-order wave equation for the sedeon wave function. It was shown that this equation can be interpreted as an equation for the potential of the baryon field. It was demonstrated that the second-order wave equation for the potentials can be represented as a system of first-order equations for the field intensities similar to the system of Maxwell's equations.

We defined the concepts of density of energy and density of energy flux for the massive baryon field. We have derived an expression that describes the energy conservation for a massive field, similar to the Poynting theorem in electrodynamics. The resulting field of point baryon is described naturally by the Yukawa potential. We have considered the interaction of two point baryons (strong interaction) due to the overlap of scalar and vector fields and calculated the baryon-baryon interaction energy as a function of the distance between them.

Assuming that the lepton field is described by the first-order wave equation, we have shown that the intensity of the lepton fields are nonzero only in the source region, so the point leptons interact only when they are at the same point in space (weak interaction).

We have demonstrated the possibility to describe the baryon-lepton interactions in terms of the overlap of scalar fields. This allows us to offer a composite model of the neutron, according to which a neutron can be considered as a bound state of a proton and an electron due to the baryon-lepton interactions. This hypothesis will be discussed in detail in our next paper.

Acknowledgements

The authors are very thankful to G.V. Mironova for kind assistance and moral support.

References

[1] H. Yukawa - On the interaction of elementary particles I, Proceedings of the Physico-Mathematical Society of Japan, **17**, 48 (1935).

[2] G.E. Brown and A.D. Jackson - "The nucleon-nucleon interaction", North-Holland Publishing Company, Amsterdam, 1976.

[3] J.L. Gammel, R.S. Christian and R.M. Thaler - Calculation of phenomenological nucleon-nucleon potentials, Physical Review, **105**, 311 (1957).

[4] P.S. Signell and R.E. Marshak - Phenomenological two-nucleon potential up to 150 Mev, Physical Review, **106**, 832 (1957).

[5] J.L. Gammel, and R.M. Thaler - Spin-orbit coupling in the proton-proton interaction, Physical Review, **107**, 291 (1957).

[6] R.J.N. Phillips - The two-nucleon interaction, Reports on Progress in Physics, **22**, 562 (1959).

[7] M. Lacombe, B. Loiseau, J.M. Richard, R. Vinh Mau, J. Cote, P. Pire and R. de Tourreil - Parameterization of the Paris N-N potential, Physical Review C, **21**, 861 (1980).

[8] R. Machleidt, K. Holinde and Ch. Elster - The bonn meson-exchange model for the nucleon-nucleon interaction, Physical Reports, **149**, 1 (1987).

[9] V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen and J.J. de Swart - Construction of high-quality NN potential models, Physical Review C, **49**, 2950 (1994).

[10] R.B. Wiringa, V.G.J. Stoks and R. Schiavilla - Accurate nucleon-nucleon potential with charge-independence breaking, Physical Review C, **51**, 38 (1995).

[11] V.I. Kukulin, V.N. Pomerantsev, A. Faessler, A.J. Buchmann and E.M. Tursunov - Moscow-type NN potentials and three-nucleon bound states, Physical Review C, **57**, 535 (1998).

[12] R. Machleidt - High-precision, charge-dependent Bonn nucleon-nucleon potential, Physical Review C, **63**, 024001 (2001).

[13] S. Weinberg - Nuclear forces from chiral lagrangians, Physics Letters B, **251**, 288 (1990).

[14] S. Weinberg - Effective chiral lagrangians for nucleon-pion interactions and nuclear forces, Nuclear Physics B, **363**, 3 (1991).

[15] C. Ordonez, L. Ray and U. van Kolck - Two-nucleon potential from chiral lagrangians, Physical Review C, **53**, 2086 (1996).

[16] E. Epelbaum, H.-W. Hammer and Ulf-G. Meissner - Modern theory of nuclear forces, Review of Modern Physics, **81**, 1773 (2009).

[17] R. Machleidt and D.R. Entem - Nuclear forces from chiral EFT: the unfinished business, Journal of Physics G: Nuclear and Particle Physics, **37**, 064041 (2010).

[18] R. Machleidt and I. Slaus - The nucleon-nucleon interaction, Journal of Physics G: Nuclear and Particle Physics, **27**, R69 (2001).

[19] A.C. Cordon and E.R. Arriola - Renormalization versus strong form factors for one-boson-exchange potentials, Physical Review C, **81**, 044002 (2010).

[20] S. Ulrych - The Poincare mass operator in terms of a hyperbolic algebra, Physics Letters B, **612**(1-2), 89 (2005).

[21] C. Cafaro and S.A. Ali - The spacetime algebra approach to massive classical electrodynamics with magnetic monopoles, Advances in Applied Clifford Algebras, **17**, 23 (2006).

[22] N. Candemir, M. Tanisli, K. Ozdas and S. Demir - Hyperbolic octonionic Proca-Maxwell equations, Zeitschrift fur Naturforschung A, **63a**, 15-18, (2008).

[23] S. Ulrych - Considerations on the hyperbolic complex Klein-Gordon equation, Journal of Mathematical Physics, **51**(6), 063510 (2010).

[24] S. Demir and M. Tanisli - A compact biquaternionic formulation of massive field equations in gravi-electromagnetism, European Physical Journal - Plus, **126**, 115 1-12 (2011).

[25] V.L. Mironov and S.V. Mironov - Octonic first-order equations of relativistic quantum mechanics, International Journal of Modern Physics A, **24**(22), 4157 (2009).

[26] R. Penney - Octonions and Dirac equation, American Journal of Physics, **36**, 871 (1968).

[27] A.J. Davies - Quaternionic Dirac equation, Physical Review D, **41**(8), 2628 (1990).

[28] S. De Leo and P. Rotelli - Quaternion scalar field, Physical Review D, **45**(2), 575 (1992).

[29] S. De Leo and K. Abdel-Khalek - Octonionic Dirac equation, Progress of Theoretical Physics, **96**, 833 (1996).

[30] W.P. Joyce - Dirac theory in spacetime algebra: I. The generalized bivector Dirac equation, Journal of Physics A: Mathematical and General, **34**, 1991 (2001).

[31] V.L. Mironov and S.V. Mironov - Noncommutative sedeons and their application in field theory, ArXiv: http://arxiv.org/abs/1111.4035 (2011).

[32] V.L. Mironov and S.V. Mironov - Reformulation of relativistic quantum mechanics equations with noncommutative sedeons, accepted in Journal of Applied Mathematics (2013).

[33] V.L. Mironov, S.V. Mironov and S.A. Korolev - Sedeonic theory of massless fields, ArXiv: http://arxiv.org/abs/1206.5969 (2012).

[34] V.L. Mironov and S.V. Mironov - Octonic representation of electromagnetic field equations, Journal of Mathematical Physics, **50**, 012901 (2009).

[35] V.L. Mironov and S.V. Mironov Octonic second-order equations of relativistic quantum mechanics, Journal of Mathematical Physics, **50**, 012302 1-13 (2009).

[36] V.L.Mironov and S.V.Mironov Sedeonic generalization of relativistic quantum mechanics, International Journal of Modern Physics A, **24**(32), 6237 (2009).